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Two Descriptions of State Spaces of Orthomodular Structures

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We present and compare two descriptions of state spaces of orthomodular structures (orthoalgebras, etc.). Both are based on families of Boolean algebras and their corresponding hypergraphs called Greechie diagrams. The first approach is based on *pasted families of Boolean algebras* and gives all orthoalgebras as their pastings. It gives a complete description of the orthoalgebra, as well as its state spaces and evaluation functionals. The second approach is based on the new notion of *semipasted families of Boolean algebras*. This structure does not correspond directly to a unique orthomodular structure, but it describes the space of states and evaluation functionals of some orthoalgebra, even an orthomodular lattice. As there is much more freedom in construction of semipasted families of Boolean algebras, they provide an efficient tool for finding orthomodular lattices with given state space properties.

1. INTRODUCTION

In a classical system, the observable events form a Boolean algebra. The states may be identified with probability measures. The logic of quantum mechanics is not distributive. For its system of events, several algebraic structures were suggested, e.g., orthomodular lattices, orthomodular posets, etc. Their study was initiated in ref. 3 and further motivated in ref. 16. Combinatorial techniques (using hypergraphs) were introduced in this field by Greechie [10] and generalized by Dichtl [6]. They became an essential tool in the theory of orthomodular structures [14, 32]. This technique was extended by further constructions, e.g., the pasting of orthomodular posets [29] and hypergraph representation of state spaces [28]. At present, combinatorial techniques for orthomodular structures are frequently used to find

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examples or to demonstrate new notions and ideas. In this paper, we collect some of them in a unified and new formulation.

The paper is organized as follows: First we define orthomodular structures and present their basic properties. Further, we introduce pasted families of Boolean algebras. We define their pastings and formulate conditions under which they are orthoalgebras, orthomodular posets, or orthomodular lattices. We use hypergraphs to describe pasted families of Boolean algebras, resp. the corresponding orthomodular structures, and we show how they determine the elements of the structure and the states. Then semipasted families of Boolean algebras are introduced as a new structure. They are again represented by hypergraphs. We derive a new type of correspondence—a functional isomorphism. It allows one to determine the states and evaluation functionals, but not the elements of the structure. Its advantage is that each hypergraph corresponds to a semipasted family of Boolean algebras and each such family represents—up to a functional isomorphism—an orthomodular lattice. This makes its use in examples very easy. An overview of applications of this technique is given at the end of the paper.

2. ORTHOMODULAR STRUCTURES

Orthoalgebras and their special cases, orthomodular posets and orthomodular lattices, were proposed and studied as possible event structures of quantum systems. Here we summarize the basic notions from the theory of orthoalgebras which will be used in the sequel.

Definition 2.1 [9]. An *orthoalgebra* is a quadruple $(L, \oplus_L, \mathbf{0}_L, \mathbf{1}_L)$, where L is a set, $\mathbf{0}_L, \mathbf{1}_L \in L$, and $\oplus_L$ is a partial binary operation on L satisfying the following properties:

(OA1) $\forall a, b \in L$: $a \oplus_L b = b \oplus_L a$
(OA2) $\forall a, b, c \in L$: $a \oplus_L (b \oplus_L c) = (a \oplus_L b) \oplus_L c$
(OA3) $\forall a \in L\, \exists!\, d \in L$: $a \oplus_L d = \mathbf{1}_L$
(OA4) $\forall a \in L$: $a \oplus_L a$ is defined iff $a = \mathbf{0}_L$

[As $\oplus_L$ is a partial operation, (OA1) and (OA2) should be read: If one side of the equality exists, then the other exists, too, and both sides are equal.]

From now on, L denotes an orthoalgebra unless stated otherwise.

For $a, b \in L$, we define $a \leq_L b$ iff there is an element $c \in L$ such that $b = a \oplus_L c$. Then $\leq_L$ is a partial order inducing partial lattice operations $\wedge_L$, $\vee_L$ on L. When we use them in expressions, we automatically assume their existence. We define a unary operation ${}'^L\colon L \to L$ assigning to each $a \in L$ the unique element d satisfying (OA3). This is an involutive antiisomorphism of L such that $a \wedge_L a'^L = \mathbf{0}_L$ for all $a \in L$. It is called an *orthocomplemen-*

tation. These operations equip L with the structure of an *orthoposet* [1, 14], but not all orthoposets are orthoalgebras.

Sometimes we speak of an orthoalgebra L instead of $(L, \oplus_L, \mathbf{0}_L, \mathbf{1}_L)$ and we omit the indices of $\oplus, \leq, \wedge, \vee, ', \mathbf{0}, \mathbf{1}$ when there is no risk of confusion.

For $a, b \in L, a \leq b$, we define the *interval* $[a, b]_L = \{c \in L\colon a \leq c \leq b\}$. An *atom* in L is an element $a \in L \setminus \{\mathbf{0}\}$ such that $[\mathbf{0}, a]_L = \{\mathbf{0}, a\}$. We denote by $\mathscr{A}(L)$ the set of all atoms of L. We say that L is *chain-finite* iff each of its chains (=linearly ordered subsets) is finite. Throughout this paper, intervals without indices are reserved for intervals of real numbers; all other intervals are indexed by the respective poset. We always consider an interval $[a, b]_L$ with the partial ordering inherited from L. We denote the bounds of posets by $\mathbf{0}$, $\mathbf{1}$ (eventually with indices, e.g., $\mathbf{0}_L$, $\mathbf{1}_L$), while the symbols 0, 1 are reserved for real numbers or constant functionals.

Two orthoalgebras $(K, \oplus_K, \mathbf{0}_K, \mathbf{1}_K)$, $(L, \oplus_L, \mathbf{0}_L, \mathbf{1}_L)$ are called *isomorphic* iff there is a surjective mapping $i\colon K \to L$ such that, for all $a, b \in K$, $i(a) \oplus_L i(b)$ exists iff $a \oplus_K b$ exists, and if this is the case, $i(a) \oplus_L i(b) = i(a \oplus_K b)$.

Two elements a, b of an orthoalgebra L are called *orthogonal* iff $a \oplus b$ is defined (in symbols $a \perp b$). This occurs iff $a \leq b'^L$.

Notice that every Boolean algebra B is an orthoalgebra if we take for $\mathbf{0}_B$, $\mathbf{1}_B$ its bounds and we define $a \oplus_B b = a \vee_B b$ whenever $a \leq_B b'^B$. In this paper, we shall understand Boolean algebras as orthoalgebras this way. We do the same for more general structures, orthomodular lattices and orthomodular posets (see refs. 1 and 14 for their standard definitions). Thus we say that an orthoalgebra L is:

- An *orthomodular poset* (*OMP*) iff each orthogonal pair has a join in L
- An *orthomodular lattice* (*OML*) iff L is a lattice
- A *Boolean algebra* (*BA*) iff L is a distributive lattice

Understanding OMPs, OMLs, and BAs this nonstandard way as orthoalgebras means that we have different operations describing the same (categorically equivalent) structure. When we want to talk about any of the three classes (OAs, OMPs, OMLs), we speak of *orthomodular structures*.

In OMPs and OMLs, $a \oplus b = a \vee b$. In orthoalgebras, this join need not exist.

A subset A of an orthoalgebra L is called a *Boolean subalgebra* iff:

- $\mathbf{0}_L \in A$
- $a \in A \Rightarrow a'^L \in A$
- $(A, \oplus_A, \mathbf{0}_L, \mathbf{1}_L)$, where $\oplus_A$ is the restriction of $\oplus_L$ to A, is a Boolean algebra.

Two elements a, b in L are called *compatible*, in symbols $a \leftrightarrow b$, iff they are contained in a Boolean subalgebra of L.

Definition 2.2. A *block* in an orthoalgebra is a maximal Boolean subalgebra.

A standard use of Zorn's Lemma implies that every orthoalgebra is a union of its blocks.

Definition 2.3. Let L be an orthoalgebra. A *state* on L is a mapping s: $L \to [0, 1]$ such that:

(SOA1) $s(\mathbf{1}) = 1$
(SOA2) $a, b \in L, a \perp b \Rightarrow s(a \oplus b) = s(a) + s(b)$

We shall define states also on other structures—orthomodular lattices, hypergraphs, etc. We always denote by $\mathcal{S}(L)$ the set of all states on L—the *state space* of L. We assume $\mathcal{S}(L) \subseteq [0, 1]^L$ with the product (=weak) topology. It is always compact and convex. The reverse implication also holds (see Theorem 9.1).

Orthoalgebras are a relatively new topic in the study of orthomodular structures. Although they are based on earlier ideas by Foulis and Randall (see, e.g., ref. 8), they became a popular area only in the late 80s. This is why they are not mentioned in the basic literature on OMLs [1, 14, 32]. In the 90s, more general structures, particularly effect algebras, were studied more intensely. We present arguments showing that orthoalgebras are exactly the most general structures allowing some basic construction techniques typical for OMPs and OMLs. Therefore they should play an important role also in the future.

3. PASTED FAMILIES OF BOOLEAN ALGEBRAS

In this section, we introduce pasted families of Boolean algebras. This structure was typical for the early studies of orthomodular structures [6, 10, 34] and it is close to the original motivation and interpretation of quantum logic. We return to it because it forms a natural link between hypergraphs and orthomodular structures.

Definition 3.1. A *pasted family of Boolean algebras* (abbr. *PF*) is a family $\mathcal{F}$ of Boolean algebras such that, for each $A, B \in \mathcal{F}$, $A \neq B$:

(PF1) $A \not\subseteq B$
(PF2) $A \cap B$ is a Boolean subalgebra of A and of B on which the operations of A, B coincide
(PF3) $\forall a \in A \cap B \; \exists C \in \mathcal{F}$: $[\mathbf{0}, a]_A \cup [\mathbf{0}, a']_B \subseteq C$

Remark 3.2. The intersection $A \cap B$ always contains the bounds $\mathbf{0}$, $\mathbf{1}$. These bounds, as well as orthocomplements, are the same in all elements of $\mathcal{F}$, so there is no need to index them by the respective Boolean algebra. The condition (PF3) is symmetric: There is also some $D \in \mathcal{F}$ containing $[\mathbf{0}, a']_A \cup [\mathbf{0}, a]_B$.

Notice that elements of a PF $\mathcal{F}$ are Boolean algebras. We often refer to elements of the union $\cup\,\mathcal{F}$ which are elements of the Boolean algebras in question (=events of the system). We use the notation $\mathcal{A}(\mathcal{F}) = \cup_{B\in\mathcal{F}}\, \mathcal{A}(B)$ and we call the elements of $\mathcal{A}(\mathcal{F})$ *atoms* of $\mathcal{F}$ (they are atoms of the Boolean algebras in $\mathcal{F}$).

Definition 3.3. Two pasted families of Boolean algebras $\mathcal{F}$ and $\mathcal{G}$ are *isomorphic* iff there is a one-to-one mapping i: $\cup\mathcal{F} \to \cup\mathcal{G}$ such that, for each $B \in \mathcal{F}$, $i|B$ is a (Boolean) isomorphism of B and $i(B)$, and $\mathcal{G} = \{i(B)\colon B \in \mathcal{F}\}$.

Definition 3.4. A pasted family of Boolean algebras $\mathcal{F}$ is *chain-finite* iff there is no infinite set $M \subseteq \cup\mathcal{F}$ such that each finite subset of M is contained in a Boolean algebra from $\mathcal{F}$.

In particular, all elements of a chain-finite PF are finite Boolean algebras.

The definition of a state on a pasted family of Boolean algebras is a canonical extension of a state on a Boolean algebra (defined in Definition 2.3).

Definition 3.5. Let $\mathcal{F}$ be a pasted family of Boolean algebras. A *state* on $\mathcal{F}$ is a mapping s: $\cup\mathcal{F} \to [0, 1]$ such that, for each $B \in \mathcal{F}$, $s|B$ is a state on B (Definition 2.3).

4. HYPERGRAPHS

Since refs. 10 and 34, hypergraphs have been used as a powerful tool for description and graphical representation of orthomodular structures. In this section, we summarize the basic notions and the relationship of hypergraphs to pasted families of Boolean algebras.

Definition 4.1. A *hypergraph* is a couple $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a nonempty set (of *vertices*) and $\mathcal{E}$ is a covering of $\mathcal{V}$ by nonempty subsets of $\mathcal{V}$ (*edges*), i.e., $\cup\mathcal{E} = \mathcal{V}$. A *state* on $\mathcal{H}$ is a mapping s: $\mathcal{V} \to [0, 1]$ such that, for each $E \in \mathcal{E}$, $\Sigma_{v\in E}\, s(v) = 1$. Two hypergraphs $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$, $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ are *isomorphic* iff there is a one-to-one mapping i: $\mathcal{V}_1 \to \mathcal{V}_2$ such that $\mathcal{E}_2 = \{i(E)\colon E \in \mathcal{E}_1\}$

The notion of a state on a hypergraph was used without explicit formulation in refs. 10 and 34 and studied in detail in ref. 11. Chain-finite pasted families of Boolean algebras allow a correspondence with hypergraphs which induces also a one-to-one correspondence between their state spaces:

Proposition 4.2. Let $\mathcal{F}$ be a chain-finite pasted family of Boolean algebras. The couple $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \mathcal{A}(\mathcal{F})$, $\mathcal{E} = \{\mathcal{A}(B)\colon B \in \mathcal{F}\}$, is a hypergraph called the *Greechie diagram* of $\mathcal{F}$. The PF $\mathcal{F}$ is determined by its Greechie diagram uniquely (up to an isomorphism). The restriction mapping h: $\mathcal{S}(\mathcal{F}) \to \mathcal{S}(\mathcal{H})$ defined by $h(s) = s|\mathcal{A}(\mathcal{F})$ is an affine homeomorphism.

We define Greechie diagrams only for chain-finite PFs. Without chain-finiteness, one may obtain hypergraphs which do not allow a unique reconstruction of the original structure of a PF and its state space. The corresponding notion in terms of hypergraphs is the following:

Definition 4.3. A hypergraph is *chain-finite* iff it does not contain an infinite set V of vertices such that each finite subset of V is contained in an edge.

Example 4.4. The following hypergraph $(\mathcal{V}, \mathcal{E})$ has finite edges, but it is not chain-finite: We take $\mathcal{V} = \{a_n, b_n, c_n: n \in N\}$, $\mathcal{E} = \{\{a_1, \ldots, a_n, b_n, c_n\}: n \in N\}$. Each finite subset of the set $V = \{a_n: n \in N\}$ is contained in an edge.

The Greechie diagrams of chain-finite PFs are chain-finite hypergraphs. There are chain-finite hypergraphs which are not Greechie diagrams of PFs. (In figures, we denote vertices of hypergraphs by small circles and edges by smooth curves.)

Example 4.5. Hypergraphs in Fig. 1 are not Greechie diagrams of PFs. The hypergraph in Fig. 1a violates the condition (PF1). In a pasting of Boolean algebras A, B, we would have $a = (c \oplus_A d)'^A = (c \oplus_B d)'^B = b$, but a, b are denoted as distinct atoms. For the same argument, the hypergraph in Fig. 1b violates (PF1). The hypergraph in Fig. 1c violates (PF2). Indeed, we obtain $a \oplus_A b = (c \oplus_A d)'^A = (c \oplus_B d)'^B = e \oplus_B f$, hence $a \oplus_A b$, $e \oplus_B f$ are the same elements in A, but $a \oplus_A b = (e \oplus_B f)'^C$, so they are complementary in C. The hypergraph in Fig. 1d violates (PF3). The element

$$i = a \oplus_A b = (c \oplus_A d)'^A = (c \oplus_B d)'^B = e \oplus_B f = (g \oplus_C h)'^C \in A \cap C$$

but there is no $D \in \mathcal{F}$ containing $[\mathbf{0}, i]_A \cup [\mathbf{0}, i'^C]_C$.

5. ORTHOALGEBRAS AS PASTINGS

In this section we shall associate with a pasted family of Boolean algebras a single algebraic structure—its pasting—which appears to be an orthoalgebra.

Definition 5.1. Let $\mathcal{F}$ be a pasted family of Boolean algebras. On $L = \bigcup\mathcal{F}$, we define the partial operation $\oplus_L$ as the union of all $\oplus_A$, $A \in \mathcal{F}$, i.e., $a \oplus_L b = c$ iff there is an $A \in \mathcal{F}$ such that $a \oplus_A b = c$. The quadruple $(L, \oplus_L, \mathbf{0}, \mathbf{1})$ is called the *pasting* of $\mathcal{F}$.

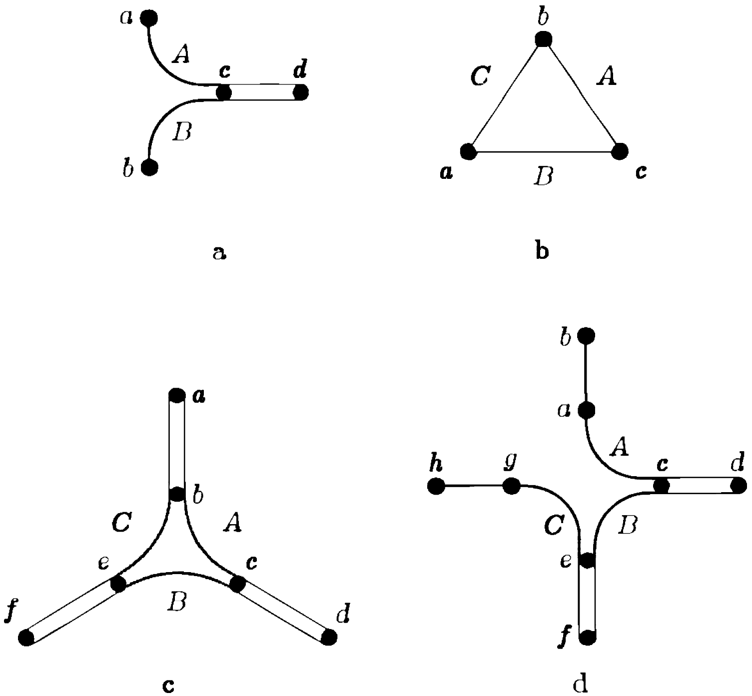

Fig. 1. Hypergraphs from Ex. 4.5.

The following proposition states that orthoalgebras are exactly pastings of PFs. Although this fact is not very difficult to prove, it seems to be explicitly formulated only in ref. 23 (for partial results, see ref. 14, §4, Prop. 13, and refs. 6 and 33).

Proposition 5.2 [23]. The pasting of a pasted family of Boolean algebras is an orthoalgebra. Conversely, every orthoalgebra is a pasting of a pasted family of Boolean algebras, namely of the family of its blocks.

Remark 5.3. The associativity of $\oplus_L$ follows from (PF3): If $(a \oplus_L b) \oplus_L c$ is defined, then there are $A, B \in \mathcal{F}$ such that $a \oplus_A b = a \oplus_L b$ and $(a \oplus_L b) \oplus_B c = (a \oplus_L b) \oplus_L c$. Applying (PF3) to $d = a \oplus_L b \in A \cap B$, we find a $C \in \mathcal{F}$ containing $[0, d]_A \cup [0, d']_B$. Thus $a, b, c \in C$ and all calculations can be made in this Boolean subalgebra:

$$(a \oplus_L b) \oplus_L c = (a \oplus_C b) \oplus_C c = a \oplus_C (b \oplus_C c) = a \oplus_L (b \oplus_L c)$$

Necessary and sufficient conditions are known for a PF to form an OMP, resp. an OML, as its pasting [6, 29]. The correspondence between OAs and PFs induces a correspondence between their state spaces:

Proposition 5.4. Let $\mathcal{F}$ be a pasted family of Boolean algebras. A function $s: \cup \mathcal{F} \to [0, 1]$ is a state on $\mathcal{F}$ iff it is a state on the pasting of $\mathcal{F}$.

Corollary 5.5. Let L be a chain-finite orthoalgebra and $\mathcal{H}$ its Greechie diagram. The restriction mapping $h: \mathcal{S}(L) \to \mathcal{S}(\mathcal{H})$ defined by $h(s) = s|\mathcal{A}(L)$ is an affine homeomorphism.

Example 5.6 [13]. The hypergraph in Fig. 2a is the Greechie diagram of an orthoalgebra. Each state on any block has a unique extension to the whole orthoalgebra. The state space is a triangle. Its analogy to the state space of the Boolean algebra 2^3 will be formulated in Section 7. The hypergraph in Fig. 2b admits only one state (evaluating each vertex to 1/3). It is the Greechie diagram of an orthoalgebra admitting exactly one state.

6. SEMIPASTED FAMILIES OF BOOLEAN ALGEBRAS

Pasted families of Boolean algebras are the basic combinatorial tool for constructions of orthomodular structures. Although they simplify the work substantially, they are still very complex in some cases. Here we introduce a new tool—semipasted families of Boolean algebras. They give us much more freedom in constructions of orthomodular structures with given state space properties.

Definition 6.1. A *semipasted family of Boolean algebras* (*SF*) is a family $\mathcal{F}$ of Boolean algebras such that, for each $A, B \in \mathcal{F}$, $A \cap B$ is an ideal in A and in B on which the orderings of A and B coincide.

As an alternative, semipasted families of Boolean algebras may be viewed as *simplicial complexes* [17].

Remark 6.2. Different Boolean algebras in an SF have the same lower bound, **0**, but—in contrast to PFs—different upper bounds.

We define *atoms* of an SF $\mathcal{F}$ just as for PFs, and we use the notation $\mathcal{A}(\mathcal{F}) = \cup_{B\in\mathcal{F}} \mathcal{A}(B)$. The *isomorphisms* of SFs and *states* on SFs are defined

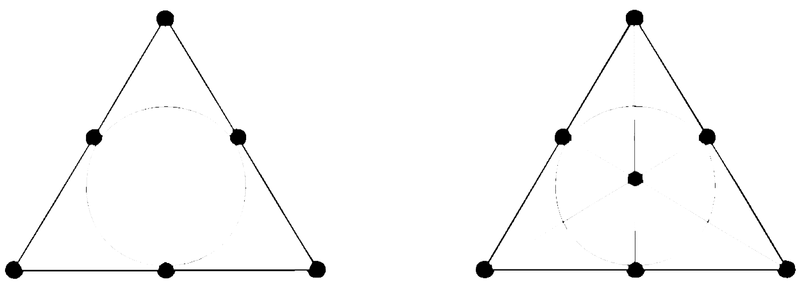

Fig. 2. Greechie diagrams of orthoalgebras from Ex. 5.6.

just as in PFs—see Defs. 3.3, 3.5. Also the definitions of a *chain-finite* SF and of a *Greechie diagram* of an SF are direct analogies of Def. 3.4 and Prop. 4.2. An SF is chain-finite iff its Greechie diagram is chain-finite. There is a one-to-one correspondence between the state space of a chain-finite SF and its Greechie diagram:

Proposition 6.3. Let $\mathcal{F}$ be a chain-finite semipasted family of Boolean algebras and $\mathcal{H}$ its Greechie diagram. Then the restriction mapping h: $\mathcal{S}(\mathcal{F}) \to \mathcal{S}(\mathcal{H})$ defined by $h(s) = s|\mathcal{A}(\mathcal{F})$ is an affine homeomorphism.

A chain-finite hypergraph may be viewed as a Greechie diagram in two ways:

1. As a Greechie diagram of a PF and also of the corresponding orthoalgebra.
2. As a Greechie diagram of an SF.

In both cases the state space remains the same. This can be easily demonstrated on a hypergraph $\mathcal{H}$ with two edges:

1. If $\mathcal{H}$ is considered as the Greechie diagram of a PF, $\{A, B\}$, then $A \cap B = I \cup I'$, where I is an ideal and $I' = \{a'\colon a \in I'\}$ is its dual filter.
2. If $\mathcal{H}$ is considered as the Greechie diagram of an SF, $\{A, B\}$, then $A \cap B = I$, where I is an ideal.

The restrictions for the state space are the same, because the value of a state s on $a' \in I'$ is uniquely determined by the value on $a \in I$; $s(a') = 1 - s(a)$.

Example 6.4. The hypergraph in Fig. 3a can be understood as the Greechie diagram of a pasted family of Boolean algebras or a semipasted family of Boolean algebras. The Hasse diagrams of its pasting as a PF (resp. as an SF) are in Fig. 3b (resp. Fig. 3c).

The advantage of SFs is more freedom in their construction—hypergraphs which are not Greechie diagrams of PFs are still Greechie diagrams of SFs:

Proposition 6.5. Every chain-finite hypergraph $(\mathcal{V}, \mathcal{E})$ is a Greechie diagram of some semipasted family of Boolean algebras, namely $\{2^E\colon E \in \mathcal{E}\}$.

Example 6.6. The hypergraphs in Figs. 1a and 1b are Greechie diagrams of SFs. The Hasse diagrams of their pastings are in Figs. 4a and 4b.

Remark 6.7. It is possible to form pastings of SFs. The resulting structure is a poset with a least element, but in general with many maximal elements. It allows us to define the *relative inverse*, and each pair of elements has a

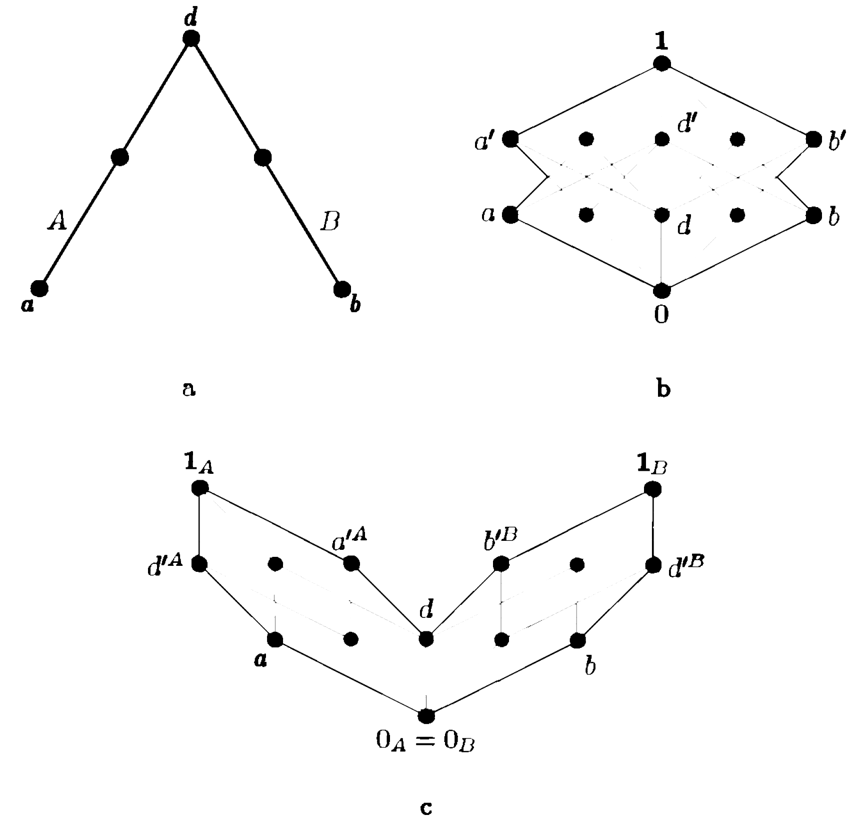

Fig. 3. A Greechie diagram (a) and the Hasse diagrams of its pasting as a PF (b), resp. SF (c); see Ex. 6.4.

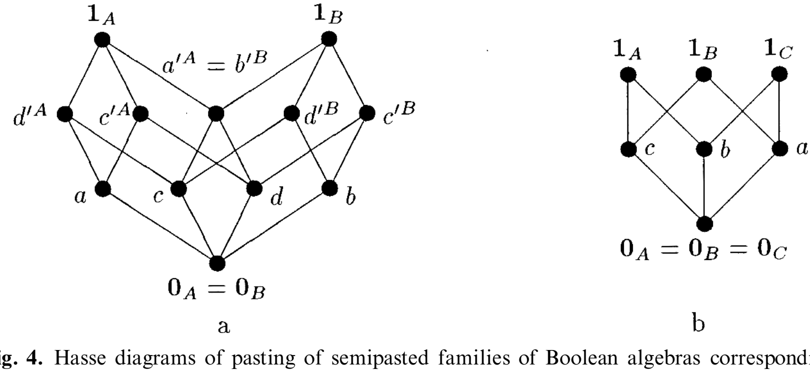

Fig. 4. Hasse diagrams of pasting of semipasted families of Boolean algebras corresponding to the Greechie diagrams in Figs. 1a, b (Ex. 6.6).

meet. Every two elements having an upper bound are compatible. Thus the pastings of SFs form a special class of *relative inverse posets* [15], but an algebraic characterization of this class is not known.

7. FUNCTIONAL EMBEDDING AND FUNCTIONAL ISOMORPHISM

In this section, we formulate the crucial notion of this paper—the correspondence between state spaces and sets of evaluation functionals called the functional embedding, resp. functional isomorphism.

Until now, we worked with event structures (OAs, PFs, SFs) and state spaces. For a structure L, states can be considered as elements of its dual, $L^* = R^L$, more exactly, of $[0, 1]^L$. There is a natural embedding $\mathbf{e}$ of L into its second dual, $L^{**} = R^{L^*}$, more exactly, into $[0, 1]^{\mathcal{S}(L)}$, defined by

$$\mathbf{e}(a)(s) = s(a) \qquad \text{for all} \quad a \in L, \quad s \in \mathcal{S}(L)$$

The functional $\mathbf{e}(a)$: $\mathcal{S}(L) \to [0, 1]$ is called the *evaluation functional* associated with a. We use the notation $\mathbf{e}(L) = \{\mathbf{e}(a): a \in L\}$.

Remark 7.1. The elements of $\mathbf{e}(L)$ are continuous affine functionals on $\mathcal{S}(L)$. The set $\mathbf{e}(L)$ is partially ordered by the usual order of real-valued functionals. There is a greatest and a least evaluation functional, namely $\mathbf{e}(\mathbf{1})$ and $\mathbf{e}(\mathbf{0})$. [These are the constant functions 1 and 0 on $\mathcal{S}(L)$.] For each evaluation functional $\mathbf{e}(a)$, its complementary functional $1 - \mathbf{e}(a)$ is the evaluation functional associated with a'. This allows us to define an "orthocomplementation" on $\mathbf{e}(L)$ by $\mathbf{e}(a)' = \mathbf{e}(a')$. The structure of $\mathbf{e}(L)$ reflects in some sense the structure of L. They coincide in the following—very important—case: An orthoalgebra L admits an *order-determining* set of states iff

$$\forall a, b \in L: \quad (a \leq b \Leftrightarrow \forall s \in \mathcal{S}(L): s(a) \leq s(b))$$

i.e.,

$$\forall a, b \in L: \quad (a \leq b \Leftrightarrow \mathbf{e}(a) \leq \mathbf{e}(b))$$

If this is the case, $\mathbf{e}(L)$ becomes an orthoalgebra isomorphic to L under the isomorphism $\mathbf{e}$. In general, an OA need not admit an order-determining set of states. A characterization of the structures obtained as the spaces of evaluation functionals of OAs is not known.

Functional embedding can be formulated in a more general context:

Definition 7.2. Let F_1 (resp. F_2) be a set of functionals on a subset S_1 (resp. S_2) of a topological linear space V_1 (resp. V_2). We call a mapping g: $F_1 \to F_2$ a *functional embedding* iff it is injective and there is an affine homeomorphism h: $S_1 \to S_2$ such that

$$[f_2 = g(f_1),\ s_2 = h(s_1)] \Rightarrow f_2(s_2) = f_1(s_1)$$

for all $f_1 \in F_1$, $s_1 \in S_1$. If, moreover, g is surjective, it is called a *functional isomorphism* and F_1, F_2 are called *functionally isomorphic*.

Functional embedding is a correspondence of sets of functionals which assumes that their domains are affinely homeomorphic. We shall apply this notion to the sets of all evaluation functionals of different structures—OAs, PFs, and SFs. Whenever K, L are two of these structures and $\mathbf{e}(K)$, $\mathbf{e}(L)$ are functionally isomorphic, we say also that K, L are functionally isomorphic. The importance of the functional isomorphism follows from the fact that it preserves many properties of state spaces, but it allows us to represent some complex structures by much simpler ones which are functionally isomorphic. The affine homeomorphism between state spaces is often a simple restriction mapping.

Example 7.3. The orthoalgebra L from Fig. 2a is functionally isomorphic to the Boolean algebra 2^3. Its evaluation functionals are $\mathbf{e}(L) = \{0, \mathbf{e}(b), \mathbf{e}(c), \mathbf{e}(d), \mathbf{e}(b'), \mathbf{e}(c'), \mathbf{e}(d'), 1\}$.

It is much more difficult to find an example analogous to Ex. 7.3 among OMPs of even OMLs. The simplest known non-Boolean OML with this property is the following:

Example 7.4 [19, 23]. Define a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, where

$$\mathcal{V} = \{a_i: i = 0, \dots, 65\}$$

$$\mathcal{E} = \{\{a_{2i}, a_{2i+1}, a_{2i+2}\}: i = 0, \dots, 32\}$$

$$\cup\ \{\{a_{2i-7}, a_{2i}, a_{2i+13}\}: i = 0, \dots, 32\}$$

(indices mod 66). It was verified by a computer that all states $s \in \mathcal{S}(\mathcal{H})$ are determined by the conditions $s(a_0) + s(a_1) + s(a_2) = 1$ and $s(a_{i+3k}) = s(a_i)$, $k = 0, \dots, 21$, $i = 0, 1, 2$. The hypergraph $\mathcal{H}$ is a Greechie diagram of an OML which is functionally isomorphic to 2^3.

Example 7.5. Let $\mathcal{V}$ be a 4-element set and $\mathcal{E}$ the collection of all 3-element subsets of $\mathcal{V}$. Each state on the hypergraph $(\mathcal{V}, \mathcal{E})$ attains 1/3 at each vertex. Analogously to Ex. 4.5, $(\mathcal{V}, \mathcal{E})$ is not a Greechie diagram of a PF and of an OA. It is the Greechie diagram of an SF (Prop. 6.5), say $\mathcal{F}$, which is functionally isomorphic to the OA from Fig. 2b, but it is much simpler. It is desirable to find an OML functionally isomorphic to $\mathcal{F}$, too. We shall do it in Th. 8.1 using a much more general tool. (In this particular case, an OML with such properties was constructed directly in ref. 22 using the idea of Ex. 7.4; it has 44 atoms. An alternative solution may be found in ref. 35.)

Remark 7.6. In ref. 28, the functional isomorphism g: $\mathbf{e}(\mathcal{A}(K)) \rightarrow \mathbf{e}(\mathcal{A}(L))$ (only evaluation functionals corresponding to atoms are considered) was introduced under the notion of *state isomorphism*. It is a stronger condition: for chain-finite orthoalgebras K, L, the functional isomorphism g allows an extension to a functional isomorphism g^*: $\mathbf{e}(K) \rightarrow \mathbf{e}(L)$, but the reverse correspondence need not exist. The advantage of the approach of ref. 28 is that it preserves more state space properties. It is applicable to a rather general class of chain-finite hypergraphs (as the representing Greechie diagrams), but not to all. A more serious disadvantage is that it does not allow an extension to structures with infinite chains. Functional isomorphism of SFs overcomes this difficulty.

8. ORTHOMODULAR LATTICES FUNCTIONALLY ISOMORPHIC TO SEMIPASTED FAMILIES OF BOOLEAN ALGEBRAS

As a main result, we construct OMLs functionally isomorphic to chain-finite SFs. This extremely simplifies the construction of OMLs with those properties of the state space which are preserved by a functional isomorphism. Instead of constructing the Greechie diagram of an OML according to refs. 6 and 10, it suffices to find a Greechie diagram of an SF (which is an arbitrary chain-finite hypergraph) and use the following theorem:

Theorem 8.1 [23]. Let $\mathcal{F}$ be a chain-finite semipasted family of Boolean algebras. Then there is an orthomodular lattice L which is functionally isomorphic to $\mathcal{F}$.

The (constructive) proof is quite technical. It uses Ex. 7.4 as an initial step. The reader is referred to ref. 23. Theorem 8.1 says that every chain-finite hypergraph represents the state space of an OML in the sense of functional isomorphism of the corresponding SF.

Example 8.2. The SF with the Greechie diagram in Fig. 1b has exactly one state and three (constant) evaluation functionals, 0, 1/2, and 1. According to Th. 8.1, there is an OML with these evaluation functionals. No direct construction of an OML with these properties seems to be described in the literature. It is not easy to find it without the use of Th. 8.1 or at least some techniques from its proof.

It is somewhat surprising that we obtained the same characterization for all three classes of orthomodular structures in question. Up to functional isomorphism, there is no distinction between OMLs, OMPs, and OAs. An explicit formulation follows (a weaker version for OMPs may be derived from ref. 28):

Corollary 8.3 (Pták's Principle [23]). Every orthoalgebra (in particular, every orthomodular poset) is functionally isomorphic to an orthomodular lattice.

Examples 5.6, and 7.4 show that the OML functionally isomorphic to an OA may be much more complex. Our technique guarantees its existence.

9. CONCLUSIONS AND OVERVIEW OF APPLICATIONS

Orthoalgebras seem to play a prominent role among several possible generalizations of orthomodular lattices. They allow generalizations of many notions, results, and techniques typical for orthomodular structures. On the other hand, they seem to be the most general structures to which we may generalize without a significant loss of important features. As an example, the notion of block is quite natural in OAs. However, the attempts to define blocks in more general orthoposets were not very successful. The same can be said about the notion of compatibility. As orthoalgebras are exactly pastings of pasted families of Boolean algebras, they are the most general structures described by Greechie diagrams.

Pastings of pasted families of Boolean algebras possess an efficient tool for constructions of orthoalgebras. The only problem is that there are requirements on their Greechie diagram which are sometimes not easy to verify. Contrary to this, the constructions with semipasted families of Boolean algebras are not limited by any restriction on the Greechie diagrams. This is allowed by weakening the correspondence between an orthomodular structure and a semipasted family of Boolean algebras to a functional isomorphism. (Instead of an exact description of the structure, we have one-to-one mappings between states and between evaluation functionals.) This simplifies many proofs and new investigations in orthomodular structures. Besides this, functional isomorphism gives us a tool for formulation of correspondences which are found in many places (e.g., refs. 2, 4). There is a limitation of this technique—it does not give orthomodular structures with order-determining sets of states.

The technique of Th. 8.1 allowed a radical simplification of the proof of the famous theorem due to Shultz:

Theorem 9.1 [21, 23, 34]. Every compact convex subset of a locally convex topological linear space is affinely homeomorphic to the state space of an orthomodular lattice.

Based on similar ideas, a characterization of spaces of σ-additive measures was found in ref. 30. Its strengthening to OMPs which are σ-*orthocomplete* (i.e., closed with respect to joins of countable orthogonal subsets) remains an open problem. Our technique enabled us to extend Th. 9.1 by proving the existence of embeddings of orthomodular posets into orthomodular posets with given state spaces, centers, and automorphism groups [13, 20, 24, 27]. The use of a functional isomorphism and Th. 8.1 allowed us also to find examples of non-Boolean OMLs which possess the Radon–Nikodým property [12, 25] and which are *fully embeddable* (see refs. 13 and 31 for the exact definition and examples). Recently, the problem of existence of σ-additive signed measures not allowing Jordan–Hahn decomposition to σ-additive positive measures was solved using Th. 8.1 [5].

There are numerous other problems in which the use of functional isomorphisms appears useful.

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